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1973 J. Phys. A: Math. Nucl. Gen. 6 1211

(http://iopscience.iop.org/0301-0015/6/8/015)

J. Phys. A: Math., Nucl. Gen., Vol. 6, August 1973. Printed in Great Britain. © 1973

Some remarks on causality, Lorentz invariance and the generalized Matthews' rule for a massive spin one vector field in an external potential

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Received 22 February 1973

Abstract. A massive spin one vector field is quantized in the presence of an external potential, with source linear in the vector field, to provide an illustration of the aspects of a quantum theory in an external potential, which are reflections of the causal nature of the corresponding classical theory. In the examples considered, the S operator of the quantum theory is found to be Lorentz invariant or not, according as the corresponding classical theory is causal or not. The normal dependence of the interaction hamiltonian, and the applicability of the generalized Matthews' rule are also discussed.

1. Introduction

The problem of determining the causal nature of the propagation of a classical field in an external potential was first studied by Velo and Zwanziger (1969a,b, 1971). It has been discussed further, in the same spirit, by Velo (1972), Shamaly and Capri (1972a,b) and Jenkins (1972). In these papers, particular examples of relativistic wave equations are considered in the presence of various external potentials. In some instances these wave equations are found to possess solutions which propagate acausally, or even fail to propagate.

Having noted the above difficulties at the classical level, it is interesting to quantize these theories and explore how their causal nature is reflected at the quantum level. Some work along these lines has been done by Johnson and Sudarshan (1961) and Schroer *et al* (1970). Now Schroer *et al* noted, following Capri (1969), that a discussion of a quantized field in an external potential, with the source linear in the field, is reducible to a discussion of the corresponding classical problem. In this case, causality or acausality, at the classical level, are reflected in the validity or breakdown of microcausality at the quantum level.

In the present paper, the interaction of a quantized massive spin one vector field with an external potential, the source being nonderivative and linear in the vector field, is discussed. The results are used to illustrate further aspects of theories, quantized in the presence of an external potential, which are reflections of the causal nature of the corresponding classical theories.

The plan of the paper is as follows. In § 2 the interaction hamiltonian in the interaction picture is calculated for the interacting vector field, and the perturbation series for the S operator is examined. In § 3 the results of § 2 are discussed and some further results quoted.

2. Interacting vector field

The most general form for the equation of a massive spin one vector field interacting with an external potential, the source being nonderivative and linear in the vector field, is given by†

$$(\partial^2+m^2)\boldsymbol{V}_\mu(x)-\partial_\mu\partial^\lambda\boldsymbol{V}_\lambda(x) = T_{\mu\lambda}(x)\boldsymbol{V}^\lambda(x) \tag{1}$$

where the potential $T_{\mu\nu}(x)$ is a second rank tensor density.

The method of Takahashi and Umezawa (see eg Takahashi 1969) is adhered to in the calculation of the interaction hamiltonian in the interaction picture. Firstly the field equation (1) is solved by the method of the Green function. Thus

$$\boldsymbol{V}_\mu(x) = V_{0\mu}(x)-\int_{-\infty}^{\infty} d_{\mu\lambda}(\partial)\,\Delta^{\mathrm{ret}}(x-x')T^{\lambda\rho}(x')\boldsymbol{V}_\rho(x')\,\mathrm{d}^4x' \tag{2}$$

where the Klein–Gordon divisor of the vector field is given by

$$d_{\mu\nu}(\partial) = g_{\mu\nu}+\frac{\partial_\mu\partial_\nu}{m^2} \tag{3}$$

and

$$\Delta^{\mathrm{ret}}(x) = \theta(x_0)\,\Delta(x) \tag{4}$$

with $\theta(x_0)$ the unit step function and $\Delta(x)$ the usual solution of the Klein–Gordon equation.

An auxiliary field is now introduced, and is defined by

$$V_\mu(x,\sigma) = V_{0\mu}(x)-\int_{-\infty}^{\sigma} d_{\mu\lambda}(\partial)\,\Delta(x-x')T^{\lambda\rho}(x')\boldsymbol{V}_\rho(x')\,\mathrm{d}^4x' \tag{5}$$

where σ is a spacelike surface, not necessarily through x. This auxiliary field satisfies free-field commutation relations, namely,

$$[V_\mu(x,\sigma),\, V_\nu^\dagger(x',\sigma)] = -\mathrm{i}\,d_{\mu\nu}(\partial)\,\Delta(x-x'). \tag{6}$$

Next, for x on the spacelike surface σ, denoted by $x|\sigma$, the expression (5) can, with the aid of the unit step function, be written with the integration range extending from $-\infty$ to $+\infty$. The resulting expression together with (2) gives

$$\boldsymbol{V}_\mu(x) = V_\mu(x|\sigma)+\int_{-\infty}^{\infty}[\theta(x_0-x_0'),\, d_{\mu\lambda}(\partial)]\,\Delta(x-x')T^{\lambda\rho}(x')\boldsymbol{V}_\rho(x')\,\mathrm{d}^4x'. \tag{7}$$

Now if n_μ is the unit normal to the spacelike surface σ, then the identities

$$[\theta(x_0-x_0'),\, g_{\mu\nu}]\,\Delta(x-x') = 0$$

$$[\theta(x_0-x_0'),\, \partial_\mu\partial_\nu]\,\Delta(x-x') = n_\mu n_\nu\,\delta^4(x-x')$$

together with (3) may be used to reduce (7) to the following form:

$$\boldsymbol{V}_\mu(x) = V_\mu(x|\sigma)+\frac{n_\mu n^\lambda}{m^2}T_{\lambda\rho}(x)\boldsymbol{V}^\rho(x). \tag{8}$$

† Heisenberg picture field operators are in bold face type, whilst their interaction picture counterparts are in light face type. The metric used is $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, $\partial_\mu = \partial/\partial x^\mu$ and the usual summation convention is assumed for Greek indices.

The next step is to solve this equation for the Heisenberg picture field in terms of the auxiliary field. Thus (8) may be easily seen to be satisfied by

$$V_\mu(x) = V_\mu(x|\sigma) + \frac{n_\mu n^\lambda T_{\lambda\rho}(x)}{m^2 - n^\alpha n^\beta T_{\alpha\beta}(x)} V^\rho(x|\sigma). \tag{9}$$

To calculate the interaction hamiltonian in the interaction picture, the following commutation relations are noted:

$$[V_\mu(x, \sigma), \mathscr{H}_{\rm int}(n, x'|\sigma)] = -\mathrm{i}\, d_{\mu\lambda}(\partial)\, \Delta(x-x') T^{\lambda\rho}(x') V_\rho(x'). \tag{10}$$

Then, when (9) has been used to express $V_\rho(x')$, on the right-hand side of (10), in terms of $V_\mu(x'|\sigma)$, the commutation relations (6) give the following form for $\mathscr{H}_{\rm int}(n, x|\sigma)$:

$$\mathscr{H}_{\rm int}(n, x|\sigma) = V_\mu^\dagger(x|\sigma) T^{\mu\nu}(x) V_\nu(x|\sigma) + V_\mu^\dagger(x|\sigma) \frac{T^{\mu\nu}(x) n_\nu n_\lambda T^{\lambda\rho}(x)}{m^2 - T^{\alpha\beta}(x) n_\alpha n_\beta} V_\rho(x|\sigma). \tag{11}$$

The interaction hamiltonian in the interaction picture $\mathscr{H}_{\rm int}(n, x)$ is finally obtained from (11) by the replacement of the auxiliary field $V_\mu(x|\sigma)$ by its interaction picture counterpart $V_\mu(x)$ etc.

$\mathscr{H}_{\rm int}(n, x)$, thus obtained, may, through Dyson's formula, be used to write the perturbation series for the S operator as

$$S = 1 - \mathrm{i} \int_{-\infty}^{\infty} T(\mathscr{H}_{\rm int}(n, x))\, \mathrm{d}^4x + \frac{(-\mathrm{i})^2}{2!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T(\mathscr{H}_{\rm int}(n, x) \mathscr{H}_{\rm int}(n, x'))\, \mathrm{d}^4x\, \mathrm{d}^4x' + \ldots.$$

Now denoting $V_\mu^\dagger(x) T^{\mu\nu}(x) V_\nu(x)$ symbolically by $V^\dagger . T . V(x)$ etc, and using Wick's theorem, here interpreted in a form, which, on defining $\theta(0) = \frac{1}{2}$, is equivalent to assuming that the interaction hamiltonian is symmetrized in the field operators $V_\mu(x)$ and $V_\mu^\dagger(x)$ (see Kvitky and Mouton 1972),

$$\begin{aligned} S = -\mathrm{i} \int_{-\infty}^{\infty} &:\overline{V^\dagger . T . V}(x): \mathrm{d}^4x - \frac{\mathrm{i}}{m^2} \int_{-\infty}^{\infty} (:V^\dagger . T . n(x) n . T . V(x): \\ &+ :\overline{V^\dagger . T . n(x) n . T . V}(x):)\, \mathrm{d}^4x - \tfrac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (:V^\dagger . T . \overline{V(x) V}^\dagger . T . V(x'): \\ &+ :\overline{V^\dagger . T . V(x) V^\dagger . T . V}(x'): + :\overline{V^\dagger . T . V}(x) V^\dagger . T . V(x'): \\ &+ :V^\dagger . T . V(x) \overline{V^\dagger . T . V}(x'): + :\overline{V^\dagger . T . \overline{V(x) V}^\dagger . T . V}(x'): \\ &+ :\overline{V^\dagger . T . V}(x) \overline{V^\dagger . T . V}(x'):)\, \mathrm{d}^4x\, \mathrm{d}^4x' + \ldots \end{aligned} \tag{12}$$

where terms of order higher than the second, and those not having noncovariant parts have been excluded. Next the identity, for the contraction of the vector field,

$$\overline{V_\mu(x) V}_\nu^\dagger(x') = -\mathrm{i}\, d_{\mu\nu}(\partial)\, \Delta_{\rm c}(x-x') - \frac{\mathrm{i}}{m^2} n_\mu n_\nu\, \delta^4(x-x') \tag{13}$$

should be noted, where

$$\Delta_{\rm c}(x) = \theta(x_0)\, \Delta^+(x) - \theta(-x_0)\, \Delta^-(x)$$

with $\Delta^\pm(x)$ being respectively the usual positive and negative frequency solutions of the Klein–Gordon equation.

Three cases are now distinguished:

(i) $n^\alpha n^\beta T_{\alpha\beta}(x) \equiv 0$.
(ii) $n^\alpha n^\beta T_{\alpha\beta}(x) \not\equiv 0$ and not proportional to $n^\alpha n_\alpha$.
(iii) $n^\alpha n^\beta T_{\alpha\beta}(x) = \psi(x) n^\alpha n_\alpha = \psi(x)$. Since n_α is a unit vector.

In case (i) the interaction hamiltonian reduces to

$$\mathscr{H}^{(i)}_{\text{int}}(n, x) = V^\dagger . T . V(x) + \frac{1}{m^2} V^\dagger . T . n(x) n . T . V(x).$$

Now on account of the identity $n^\alpha n^\beta T_{\alpha\beta}(x) \equiv 0$ and the expression (13), for the contraction of the vector field, the noncovariant parts of the 1st, 6th, 7th and 9th terms of (12) vanish identically; whilst there is a cancellation of the noncovariant parts in the 2nd, 4th, 5th and 3rd, 8th terms of (12), separately. Thus to second order all the noncovariant parts of the S operator disappear, leaving it Lorentz invariant. The Lorentz invariance of the S operator in all orders of perturbation theory may be verified by a tedious combinatorial argument, which is omitted; and it is found that the effective interaction hamiltonian

$$\mathscr{H}^{\text{eff}}_{\text{int}}(x) = -\mathscr{L}_{\text{int}}(x) = V^\dagger . T . V(x)$$

may be used in conjunction with an effective propagator for the vector field, just comprising the covariant part of (13), to generate the S operator for case (i). This is the generalized Matthews' rule (Matthews 1949, Takahashi 1969), which has been seen to be valid for case (i).

In case (ii) there is no simplification of the form (11) for the interaction hamiltonian. Further, whilst the cancellation of the noncovariant parts in the 2nd, 4th and 5th terms of (12) still occurs in this case, the similar cancellation between the 3rd and 8th terms is not complete. In addition the 1st, 6th, 7th and 9th terms now have nonvanishing noncovariant parts, which do not cancel amongst themselves. Thus in both first and second orders the S operator has noncovariant parts. The S operator is thus not a Lorentz invariant quantity, and the generalized Matthews' rule is violated.

In case (iii) the interaction hamiltonian reduces to

$$\mathscr{H}^{(iii)}_{\text{int}}(n, x) = V^\dagger . T . V(x) + \frac{V^\dagger . T . n(x) n . T . V(x)}{m^2 - \psi(x)}. \tag{14}$$

On account of $n^\alpha n^\beta T_{\alpha\beta}(x) = \psi(x)$ being independent of n_μ, and the expression (13), for the vector field propagator, the 1st, 6th, 7th and 9th terms of (12) contain no noncovariant parts. In addition, there is a cancellation between the noncovariant parts in the 2nd, 4th and 5th terms, and the 3rd and 8th terms, separately. However, as is evidenced by the identity

$$\overline{V^\dagger . T . V}(x) = -\mathrm{i}\, T_{\lambda\rho}(x)\, d^{\rho\lambda}(\partial)\, \Delta_c(0) - \frac{\mathrm{i}}{m^2} \psi(x)\, \delta^4(0),$$

a use of the effective interaction hamiltonian

$$\mathscr{H}^{\text{eff}}_{\text{int}}(x) = -\mathscr{L}_{\text{int}}(x) = V^\dagger . T . V(x)$$

together with the covariant part of (13), as the effective vector field propagator, generates an S operator, which differs from that of case (iii) by a Lorentz invariant function of $\delta^4(0)$. A tedious combinatorial argument, which is omitted, may be used to extend this result to all orders of perturbation theory; when it is seen that, whilst the S operator is Lorentz invariant, Matthews' rule is violated.

Finally it should be noted that Velo and Zwanziger (1969b) have shown, at the classical level, that case (i) is causal and case (ii) is not, whilst the fact, that case (iii) is causal, is implicit in their calculation for a symmetric second rank tensor external potential.

3. Discussions and conclusions

The theories considered in the previous section fall into three categories, (*a*), (*b*), (*c*), which are characterized by the following sets of properties.

(*a*) The classical theory is causal. In the corresponding quantized theory, $\mathscr{H}_{\text{int}}(n, x)$ is polynomial in both n_μ and the coupling to the external potential, the generalized Matthews' rule applies and the S operator is Lorentz invariant.

(*b*) The classical theory is causal. In the corresponding quantized theory, $\mathscr{H}_{\text{int}}(n, x)$ is polynomial in n_μ but not in the coupling to the external potential, the generalized Matthews' rule does not apply, however the S operator is Lorentz invariant.

(*c*) The classical theory is acausal. In the corresponding quantized theory, $\mathscr{H}_{\text{int}}(n, x)$ is nonpolynomial in both n_μ and the coupling to the external potential, the generalized Matthews' rule does not apply and the S operator is not Lorentz invariant.

At this stage, it should be asked, if the above categorization is of wider validity. This possibility may be examined by relaxing the restriction to nonderivative sources, and allowing a consideration of sources which are linear in the vector field and its first derivatives. Exactly the same methods as those used in § 2 may be used for the discussion, and some results are merely quoted.

It is found that the vector field with arbitrary magnetic dipole moment, which is given by (1), with $T_{\mu\nu}(x)$ proportional to the electromagnetic field-strength tensor, $F_{\mu\nu}(x)$, and with the replacement $\partial_\mu \rightarrow \pi_\mu \equiv \partial_\mu + ie\, A_\mu(x)$, $A_\mu(x)$ the electromagnetic potential, falls into category (*a*). On the other hand, if an arbitrary electric quadrupole moment is included, by adding a nonvanishing term proportional to

$$\partial_\mu F_{\lambda\rho}(x)(\pi^\lambda V^\rho(x) - \pi^\rho V^\lambda(x)) + 2\pi^\nu(\partial_\lambda F_{\mu\nu}(x) V^\lambda(x))$$

to the right-hand side of the field equation for the previous case, the resulting theory falls into category (*c*). The causal nature of the corresponding classical theories has been established by Velo and Zwanziger (1969b).

In conclusion, the following remarks may be made.

In all the examples considered here, the quantum theory, corresponding to a classical theory, which is causal in the sense of Velo and Zwanziger (1969a, b, 1971), is given in terms of an interaction hamiltonian, $\mathscr{H}_{\text{int}}(n, x)$, which is polynomial in n_μ and which leads, through Dyson's formula, to a Lorentz invariant S operator. On the other hand, the quantum theory, corresponding to an acausal classical theory, is, in all the examples considered here, given in terms of an interaction hamiltonian, which is nonpolynomial in n_μ and leads to an S operator which is not Lorentz invariant. There is no such intimate connection, between the causal nature of the classical theory, and the applicability of the generalized Matthews' rule in the corresponding quantum theory, as is evidenced by the results of § 2.

Finally, the above connections, between the causal nature of the classical theory and the Lorentz transformation properties of the S operator in the corresponding quantum theory, and the former and the normal dependence of the interaction hamiltonian in the corresponding quantum theory, are expected to remain valid for spins other than one, and when the source is only restricted to being polynomial in the quantized field and its derivatives.

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